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A new view of $d = 7$ Clifford algebra

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Abstract. A new view of $d = 7$ Clifford algebra is described in which, starting from a septet of mutually anticommuting Clifford operators, one introduces a septet of mutually commuting involutions $\in SO(8)$. These involutions can be seen to arise naturally from a study of ternary vector cross products in eight dimensions.

1. Introduction

In a recent study by the author of ternary vector cross products on eight-dimensional space [1] it emerged that certain involutions $\in SO(8)$ played an important role. As an offshoot of this study a new way of viewing $d = 7$ Clifford algebra was arrived at. In fact this new view (see equations (22)-(24) below) can be obtained quite directly, without explicit appeal to ternary vector cross proctucts, or for that matter to octonions, as will be set out in the main part of this paper. The link-up with ternary vector cross products is briefly described in the appendix.

To set the stage let us first of all describe the usual view of the SO(7) Clifford algebra associated with a real inner product space R^7. If, in an irreducible representation, the pth element b_p of an orthonomal basis for R^7 is represented by Γ_p, then in addition to

$$\Gamma_p\Gamma_q + \Gamma_q\Gamma_p = -2\delta_{pq} \tag{1}$$

we must have, by Schur's lemma,

$$(\Gamma_0 \equiv)\Gamma_1\Gamma_2 \ldots \Gamma_7 = \eta I \qquad \eta = +1 \text{ or } -1 \tag{2}$$

(since $(\Gamma_0)^2 = +I$ and Γ_0 commutes with each Γ_p). Consider the finite group G_{128} of order 128:

$$G_{128} = \{\pm I, \pm\Gamma_p, \pm\Gamma_{pq}, \pm\Gamma_{pqr}\} \tag{3}$$

where $\Gamma_{pq} = \Gamma_{[p}\Gamma_{q]}$, $\Gamma_{pqr} = \Gamma_{[p}\Gamma_q\Gamma_{r]}$. Now the 56 elements $\pm\Gamma_p, \pm\Gamma_{pq}$ have squares equal to $-I$, and the remaining 72 elements $\pm I, \pm\Gamma_{pqr}$ have squares equal to $+I$. Thus the average of these squares is $+\frac{1}{8}I$. This fact immediately entails two things (see, for example, [2] equation (45)): firstly the representation is eight dimensional over $\mathbb{C}$ and secondly it is of (Frobenius-Schur) real type. Consequently we can take the Dirac operators Γ_p to act upon a real eight-dimensional vector space E. (Information concerning Clifford algebras for spaces of other dimensions and signatures can be obtained similarly.) By averaging over G_{128} we can construct an inner product $\langle , \rangle$ for E which is invariant, that is $\Gamma_p \in O(E)$; consequently $\Gamma_p \in Sk(E, E)$ (the space of skew

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adjoint maps $E \to E$). On dimensional grounds we now arrive at the following well known results for the 63-dimensional space $L_0(E, E) \simeq \mathrm{sl}(8; \mathbb{R})$ consisting of the zero-trace linear operators on E. The $(7+21=)28$ elements

$$\{\Gamma_p\} \cup \{\Gamma_{pq}\,;\, p<q\} \tag{4}$$

form a basis for the subspace $\mathrm{Sk}(E, E) \simeq \mathrm{so}(8)$, and the 35 elements

$$\{\Gamma_{pqr}\,;\, p<q<r\} \tag{5}$$

form a basis for the subspace $S_0(E, E)$ of zero-trace self-adjoint operators.

2. The new view

As a prelude to the new view of things, consider the finite projective plane geometry consisting of the seven points

$$\mathscr{P} = \{1, 2, 3, 0', 1', 2', 3'\} \tag{6}$$

and the seven lines

$$\mathscr{L} = \{\omega, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \tag{7}$$

where

$$\begin{array}{llll} \omega = 123 & \alpha_1 = 13'2' & \alpha_2 = 21'3' & \alpha_3 = 32'1' \\ & \beta_1 = 0'11' & \beta_2 = 0'22' & \beta_3 = 0'33'. \end{array} \tag{8}$$

Thus each line consists of precisely three points and each point lies on precisely three lines. (The labelling is chosen so as to link up with [1].) The lines $\lambda \in \mathscr{L}$ together with an identity element ι form an Abelian group L_8 of order eight, upon defining

$$\begin{array}{ll} \lambda^2 = \iota & \lambda \in \mathscr{L} \\ \lambda \cdot \mu = \nu & \text{whenever } \lambda, \mu, \nu \text{ are distinct and concurrent.} \end{array} \tag{9}$$

One easily sees that this 'group of lines' L_8 is isomorphic to $\mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2$, any three non-concurrent lines (for example, $\alpha_1, \alpha_2, \alpha_3$) generating a suitable triple of Z_2 subgroups. Of course the points $p \in \mathscr{P}$ together with an identity element p_0 form an Abelian 'group of points' $\mathrm{P}_8 \simeq \mathrm{Z}_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2$, with relations

$$\begin{array}{ll} p^2 = p_0 & p \in \mathscr{P} \\ p \cdot q = r & \text{whenever } p, q, r \text{ are distinct and collinear.} \end{array} \tag{9'}$$

We may view P_8 as the dual group $\hat{\mathrm{L}}_8$ of L_8, consisting of the eight simple characters of L_8:

$$\mathrm{P}_8 \simeq \hat{\mathrm{L}}_8 = \{\chi_0, \chi_p\,;\, p \in \mathscr{P}\}. \tag{10}$$

Here χ_0 denotes the trivial character and $\chi_p(\lambda) = \varepsilon_p^\lambda$, where

$$\varepsilon_p^\lambda = \begin{cases} +1 & \text{if } p \in \lambda \\ -1 & \text{if } p \notin \lambda. \end{cases} \tag{11}$$

One easily checks that the correspondence $p_0 \leftrightarrow \chi_{p_0} \equiv \chi_0,\ p \leftrightarrow \chi_p$ establishes an isomorphism $\mathrm{P}_8 \simeq \hat{\mathrm{L}}_8$:

$$\chi_a \chi_b = \chi_{a \cdot b} \qquad a, b, \in \mathrm{P}_8. \tag{12}$$

For the purpose of applications to $d = 7$ Clifford algebra, three cyclic orderings *pqr, qrp, rpq* of the points of a line $\lambda \in \mathscr{L}$ will be considered to be 'positive', and the other three orderings 'negative'. The orderings listed in (8) are laid down as positive. We now make some choice of one-to-one correspondence $p \leftrightarrow \Gamma_p$ between $\mathscr{P}$ and the previous (but now relabelled) mutually anticommuting septet $\{\Gamma_p\}$ of Clifford operators. Any choice will do, except that, for the moment, we will restrict attention to a set $\{\Gamma_p;\ p \in \mathscr{P}\}$ of Clifford operators which are 'right-handed' in the sense that

$$(\Gamma_0 \equiv)\Gamma_{1230'1'2'3'} = +I. \tag{13}$$

For each $\lambda \in \mathscr{L}$ we define Π^λ by

$$\Gamma_{pqr} = -\Pi^\lambda \tag{14}$$

where *pqr* are the points of λ in positive order. Thus in detail we have

$$\Pi^\omega = -\Gamma_{123} \qquad \Pi^{\alpha_i} = -\Gamma_{ik'j'} \qquad \Pi^{\beta_i} = -\Gamma_{0'ii'} \tag{15}$$

where *ijk* runs through the values 123, 231, 312. Observe that

$$\begin{aligned} &\text{(i)} && (\Pi^\lambda)^2 = I && \lambda \in \mathscr{L} \\ &\text{(ii)} && \Pi^\lambda \Pi^\mu = \Pi^\nu && \text{whenever } \lambda, \mu, \nu \text{ are distinct and concurrent.} \end{aligned} \tag{16}$$

Thus $\iota \mapsto I$, $\lambda \mapsto \Pi^\lambda$ defines an eight-dimensional faithful representation, Π say, of the group of lines L_8. Notice further that Γ_p commutes or anticommutes with Π^λ according as $p \in \lambda$ or $p \notin \lambda$:

$$\Gamma_p \Pi^\lambda = \varepsilon_p^\lambda \Pi^\lambda \Gamma_p. \tag{17}$$

The usual way of dealing with the algebra

$$L(E, E) = \mathrm{Sk}(E, E) \oplus S_0(E, E) \oplus \langle I \rangle \tag{18}$$

of linear operators on E is to employ the bases (4) and (5) for $\mathrm{Sk}(E, E)$, $S_0(E, E)$ in conjunction with the multiplicative properties (1) and (13). These same bases are now perceived in a new light:

$$(\mathrm{so}(8) \simeq)\ \mathrm{Sk}(E, E) = V^7 \oplus V^{21} \qquad S_0(E, E) = W^7 \oplus W^{28} \tag{19}$$

where

$$\begin{aligned} &(R^7 \simeq) V^7 = \langle \Gamma_p;\ p \in \mathscr{P} \rangle \\ &(\mathrm{spin}(7) \simeq) V^{21} = \langle \Gamma_p \Pi^\lambda;\ p \in \mathscr{P}, \lambda \in \mathscr{L}, p \in \lambda \rangle \end{aligned} \tag{20}$$

and

$$\begin{aligned} &W^7 = \langle \Pi^\lambda;\ \lambda \in \mathscr{L} \rangle \\ &W^{28} = \langle \Gamma_p \Pi^\lambda;\ p \in \mathscr{P}, \lambda \in \mathscr{L}, p \notin \lambda \rangle. \end{aligned} \tag{21}$$

Moreover the content of $d = 7$ Clifford algebra can be summarised by the following multiplicative properties:

$$(\Gamma_p)^2 = -I \qquad (\Pi^\lambda)^2 = +I \tag{22}$$

$$\Gamma_p \Gamma_q = -\Gamma_q \Gamma_p = \Pi^\lambda \Gamma_r \qquad \Pi^\lambda \Pi^\mu = \Pi^\mu \Pi^\lambda = \Pi^\nu \tag{23}$$

$$\Gamma_p \Pi^\lambda = \begin{cases} +\Pi^\lambda \Gamma_p & \text{if } p \in \lambda \\ -\Pi^\lambda \Gamma_p & \text{if } p \notin \lambda \end{cases} \tag{24}$$

where in (23) $p \neq q$, $\lambda \neq \mu$, pqr are the points of λ in positive order and $\nu(\neq \lambda, \mu)$ is concurrent with λ, μ. There is no need to add (13) to the list (22)-(24), since, for example, the particular case $\Pi^{\beta_1}\Pi^{\beta_2}\Pi^{\beta_3} = I$ of (23) becomes in conventional notation $(-\Gamma_{0'11'})(-\Gamma_{0'22'})(-\Gamma_{0'33'}) = I$ from which (13) follows.

Incidentally we did not set out in (22)-(24) to display a minimal list of properties. For example the anticommutativity of Γ_p, Γ_q, for $p \neq q$, follows from $\Gamma_p\Gamma_q = \Pi^\lambda\Gamma_r$ upon taking inverses and using (22) and (24).

3. Canonical axes and simultaneous canonical forms

In our finite plane there is of course complete duality between points and lines. Clearly this does not go through as far as our algebra is concerned. For a start the points are associated with a septet $\{\Gamma_p\}$ of mutually anticommuting imaginary units $\in \mathrm{SO}(E)$ while the lines are associated with the septet $\{\Pi^\lambda\}$ of mutually commuting involutions $\in \mathrm{SO}(E)$. Being an involution, Π^λ is diagonalisable upon E; its $+1$ and -1 eigenspaces have dimension four, since $\mathrm{Tr}(\Pi^\lambda) = 0$. Indeed, since the seven Π^λ mutually commute, they are simultaneously diagonalisable, and their ± 1 eigenspaces are consequently mutually compatible (see, e.g., [3]). Rather than proceed, in this manner, entirely in terms of elementary linear algebra, it will speed matters if we employ some (equally elementary) group representation theory.

By averaging over the group L_8 we can construct out of the representation Π of L_8 eight projection operations F_a associated with the eight irreducible representations χ_a of L_8:

$$F_a = \mathrm{Av}_{g \in \mathrm{L}_8}\{\chi_a(g)\Pi(g)\} \qquad a \in \mathrm{P}_8 \tag{25}$$

(with $F_{p_0} \equiv F_0$). Since $\mathrm{Tr}\,(F_a) = 1$, each irreducible representation χ_a is contained precisely once in a complete reduction of Π. Consequently there exists an orthonormal basis $\{e_a\}$ for E such that $\Pi(g)e_a = \chi_a(g)e_a$ and which therefore simultaneously diagonalises all of the Π^λ in the manner

$$\Pi^\lambda e_0 = e_0 \qquad \Pi^\lambda e_p = \varepsilon_p^\lambda e_p \qquad p \in \mathscr{P}. \tag{26}$$

Now it follows from (17) that $\Pi^\lambda(\Gamma_p e_0) = \varepsilon_p^\lambda(\Gamma_p e_0)$, whence $\Gamma_p e_0 = \pm e_p$. Let us agree to fix the relative signs of the basis vectors e_a by the convention

$$\Gamma_p e_0 = +e_p \qquad p \in \mathscr{P}. \tag{27}$$

We have thereby obtained a simultaneous canonical form for all of the 64 Clifford operators of our basis for $L(E, E)$, in which the seven elements $\Pi^\lambda \in W^7$ take the diagonal form (26) and the seven operators $\Gamma_p \in V^7$ satisfy

$$\Gamma_p e_q = e_r \tag{28}$$

whenever the distinct points p, q, r are collinear and in positive order. For example, we have for Γ_1 the canonical form

$$\Gamma_1 = J_{01} + J_{23} + J_{3'2'} + J_{1'0'} \tag{29}$$

where $J_{ab} = -2e_a \wedge e_b$ (on identifying $\wedge^2 E$ with $\mathrm{Sk}(E, E)$ in the usual manner), and for Γ_{23} the canonical form

$$\Gamma_{23} = \Gamma_1\Pi^\omega = J_{01} + J_{23} - J_{3'2'} - J_{1'0'} \tag{29'}$$

(since by (26) Π^ω acts as $+1$ upon e_0, e_1, e_2, e_3 and as -1 upon e_0', e_1', e_2', e_3').

Starting from a choice of a septet of mutually orthogonal axes $<\Gamma_p>$ for V^7, together with a choice of one-to-one correspondence of the Γ_p with the points $p \in \mathscr{P}$, we have constructed an octet of mutually orthogonal axes $<e_a>$ for the space E. At first sight it appears that the axis $<e_0>$ stands out as privileged. For, leaving aside the slight privilege accorded to the vector e_0, as compared to the vectors e_p, in our convention (27), is not the axis $<e_0>$ singled out by the fact that, in the decomposition of the representation Π of L_8, it alone is associated with the trivial character χ_0 of L_8? In fact this is a spurious view of the situation which has arisen because we have dealt with the non-invariant subgroup $\mathrm{G}_8 = \Pi(\mathrm{L}_8)$ of G_{128}:

$$\mathrm{G}_8 = \{I, \Pi^\lambda;\ \lambda \in \mathscr{L}\} \subset \mathrm{G}_{128}. \tag{30}$$

To see clearly that democracy does in fact reign amongst the eight axes $<e_a>$ let us deal instead with the invariant subgroup

$$\mathrm{G}_{16} = \{\pm I, \pm\Pi^\lambda;\ \lambda \in \mathscr{L}\} \lhd \mathrm{G}_{128}. \tag{31}$$

Note that G_{128} has coset decomposition

$$\mathrm{G}_{128} = \cup_{a \in \mathrm{P}_8}(\Gamma_a \mathrm{G}_{16}) \tag{32}$$

where $\Gamma_{p_0} \equiv \Gamma_0 = I$, and that from (23)

$$\Gamma_a(\Gamma_b \mathrm{G}_{16}) = \Gamma_{a \cdot b}\mathrm{G}_{16} \qquad a, b \in P_8 \tag{33}$$

so that $\mathrm{G}_{128}/\mathrm{G}_{16}$ is isomorphic to P_8. Of course we have $\mathrm{G}_{16} \simeq \mathrm{Z}_2 \times \mathrm{G}_8$, where $\mathrm{Z}_2 = \{I, -I\} \subset \mathrm{G}_{16}$ denotes the centre of G_{128}. But equally we have $\mathrm{G}_{16} \simeq \mathrm{Z}_2 \times \mathrm{G}_8(p)$ for each $p \in \mathscr{P}$, where $\mathrm{G}_8(p) \subset \mathrm{G}_{16}$ is the conjugate of G_8 in G_{128} by Γ_p:

$$\mathrm{G}_8(p) = \Gamma_p \mathrm{G}_8 (\Gamma_p)^{-1} \tag{34}$$

Consider the dual group $\hat{\mathrm{G}}_{16}$ (consisting of the 16 simple characters of the Abelian group G_{16}) which is a G_{128} space under the action $\chi \to g\chi$ where

$$(g\chi)(n) = \chi(g^{-1}ng) \qquad g \in \mathrm{G}_{128},\ n \in \mathrm{G}_{16}. \tag{35}$$

Now G_{16} comes to us in terms of the eight-dimensional defining representation (31) in which Z_2 is represented faithfully. So we are concerned solely with those eight simple characters ε_a, $a \in \mathrm{P}_8$, given by

$$\varepsilon_a = \varepsilon \times \chi_a \qquad \textit{qua}\ \mathrm{G}_{16} \text{ as } \mathrm{Z}_2 \times \mathrm{G}_8 \tag{36}$$

where $\varepsilon(\pm I) = \pm 1$. These form a single G_{128} orbit, since we have

$$\Gamma_a \varepsilon_b = \varepsilon_{a \cdot b} \qquad a, b \in \mathrm{P}_8. \tag{37}$$

Clearly democracy reigns, the octet $\{\varepsilon_a\,;\ a \in \mathrm{P}_8\}$ forming a homogeneous space for G_{128} under the action (35). (Each of the other simple characters, obtained by replacing ε in (36) by the trivial character for Z_2, form singleton G_{128} orbits.) Of course it is merely our labelling convention which is geared to the view of G_{16} as $\mathrm{Z}_2 \times \mathrm{G}_8$. If for $p \in \mathscr{P}$ we define

$$\varepsilon_a^{(p)} = \varepsilon \times \chi_a \qquad \textit{qua}\ \mathrm{G}_{16} \text{ as } \mathrm{Z}_2 \times \mathrm{G}_8(p) \tag{36$'$}$$

then the eight characters $\varepsilon_a^{(p)}\ (= \varepsilon_{p \cdot a})$ would just be a permutation of the previous eight characters ε_a. Just as $\varepsilon_0(g) = +1$ for $g \in \mathrm{G}_8$, so does $\varepsilon_0^{(p)}(g) = +1$ for $g \in \mathrm{G}_8(p)$.

It probably clarifies things still further if we describe the foregoing set-up in terms of induced representations. Let U denote the defining eight-dimensional irreducible representation of the group G_{128}: $U(g)=g$, $g \in G_{128}$. Since the Γ_a permute the axes $<e_a>$ amongst themselves by way of

$$\Gamma_a <e_b> = <e_{a \cdot b}> \qquad a, b \in P_8 \tag{38}$$

we see that the octet $\{<e_a>; a \in P_8\}$ of canonical axes forms a transitive system of imprimitivity for the representation U. Now the isotropy group of each axis $<e_a>$ is clearly G_{16}, and the restriction to $<e_a>$ of the subduced representation $U \downarrow G_{16}$ is given by

$$U(n)e_a = \varepsilon_a(n)e_a \qquad n \in G_{16}. \tag{39}$$

Consequently we see, for each $a \in P_8$, that U can be viewed as the induced representation

$$U \simeq \varepsilon_a(G_{16}) \uparrow G_{128}. \tag{40}$$

Of course we have just been dealing with a particular case of the well known theorem due to Clifford concerning representations induced in an invariant subgroup (see [4] and, for example, [5]). In our case there were several special features. In particular the invariant subgroup G_{16} coincided with the isotropy subgroup of one (indeed every) irreducible consistuent of $U \downarrow G_{16}$, and the common multiplicity of these irreducible constituents, the set $\{\varepsilon_a\}$ of mutually conjugate representations, was 1.

4. Miscellaneous remarks

(a) Just as the septet $\{\Gamma_p ; p \in \mathscr{P}\}$ forms a maximal anticommuting subset of the basis (4), (5) for L_0 (E, E), so does the septet $\{\Pi^\lambda ; \lambda \in \mathscr{L}\}$ form a maximal commuting set.

(b) Of course, the fact that our canonical octet of axes $\{<e_a>\}$ forms a transitive system of imprimitivity for U is equally well seen in terms of the property

$$\Gamma_a F_b \Gamma_a^{-1} = F_{a \cdot b} \qquad a, b \in P_8 \tag{41}$$

of the associated family $\{F_a\}$ of projections. Incidentally these latter can be defined in terms of averaging over G_{16}:

$$F_a = \mathrm{Av}_{n \in G_{16}}\{\varepsilon_a(n)n\} \qquad a \in P_8. \tag{42}$$

Each F_a can be written as a product of three commuting projections onto (mutually compatible) four-dimensional subspaces, relative to a choice of three generators for the group $G_8(a)$. For example, choosing Π^{α_1}, Π^{α_2}, Π^{α_3} as generators of G_8, we have†

$$F_0 = \tfrac{1}{2}(I+\Pi^{\alpha_1})\tfrac{1}{2}(I+\Pi^{\alpha_2})\tfrac{1}{2}(I+\Pi^{\alpha_3}). \tag{43}$$

(c) To highlight the symmetry present in many of our considerations it helps (cf [1]) to put the octet

$$\{\varepsilon_a ; a = 0, 1, 2, 3, 0', 1', 2', 3'\} \tag{44}$$

† Since completing the present work the author has discovered, after correspondence with L P Horwitz, that essentially the same projections as the F_a were employed previously by Goldstine and Horwitz [6].

in correspondence with the eight points $\{0, 1, 2, 3, 0', 1', 2', 3'\}$ of the three-dimensional affine geometry $\mathscr{A}$ over the finite field F_2 of order 2. Points p, q, r in the previous plane projective geometry (also over F_2) are collinear if and only if the four points 0, p, q, r of the affine geometry are coplanar (the remaining four points forming a parallel plane). Each affine line consists of two points, and we have that

$$ab \text{ is parallel to } cd \Leftrightarrow \varepsilon_a\varepsilon_b = \varepsilon_c\varepsilon_d. \tag{45}$$

In the affine geometry there are seven quadrupoles of mutually parallel lines, namely

$$\begin{array}{llll} 01 & 23 & 3'2' & 1'0' \\ 02 & 31 & 1'3' & 2'0' \\ 03 & 12 & 2'1' & 3'0' \\ 00' & 11' & 22' & 33' \\ 01' & 3'2 & 32' & 0'1 \\ 02' & 1'3 & 13' & 0'2 \\ 03' & 2'1 & 21' & 0'3 \end{array} \tag{46}$$

and seven pairs $(P^\lambda, P^{\lambda *})$, $\lambda \in \mathscr{L}$, of parallel planes:

$$\begin{array}{lllllll} \lambda = \omega & \beta_1 & \beta_2 & \beta_3 & \alpha_1 & \alpha_2 & \alpha_3 \\ P^\lambda = 0123 & 011'0' & 022'0' & 033'0' & 013'2' & 021'3' & 032'1' \\ P^{\lambda *} = 0'1'2'3' & 233'2' & 311'3' & 122'1' & 0'1'32 & 0'2'13 & 0'3'21. \end{array} \tag{47}$$

For the purposes of displaying canonical forms, as embarked upon in (29), a further refinement is necessary: the two orderings ab and ba of the points of a line will be distinguished, and we will view the lines of a quadrupole in (46) as 'strictly parallel' (rather than 'antiparallel'), which we write '$\sim$', when the order is displayed. So, for example, we have

$$00' \sim 11' \sim 22' \sim 33'. \tag{48}$$

The scheme (46) is consistent with '$\sim$' being an equivalence relation which obeys the following rule:

$$\text{if } a, b, c, d \text{ are distinct, then } ab \sim cd \text{ implies } ac \sim db. \tag{49}$$

In fact, using this rule, the whole scheme (46) follows from, for example, (48) together with, for example, $01 \sim 23$. The other canonical forms analogous to (29) are now simply read off. For example, we have

$$\Gamma_{0'} = J_{00'} + J_{11'} + J_{22'} + J_{33'}. \tag{50}$$

Incidentally we have ordered the points a, b, c, d in each plane of (47) so that they satisfy $ab \sim cd$.

Since $\{\Gamma_p ; p \in \mathscr{P}\}$ is a basis for V^7, and since V^{21} is the orthogonal complement of V^7 in Sk (E, E) then for $F \in \mathrm{Sk}(E, E)$ we have $F \in V^7$ if and only if

$$F = \sum_p \lambda_p \Gamma_p \qquad \text{some } \lambda_p \in \mathbb{R} \tag{51}$$

and $F \in V^{21}$ if and only if

$$\langle F, \Gamma_p \rangle = 0 \qquad \text{each } p \in \mathscr{P}. \tag{52}$$

Using the canonical form for the Γ_p exemplified by (50) the condition (51) on F breaks down into seven sets of four equations, one set being exemplified by

$$F_{00'} = F_{11'} = F_{22'} = F_{33'} \tag{53}$$

and the seven equations in the condition (52) are exemplified by

$$\langle F, \Gamma_{0'}\rangle = F_{00'} + F_{11'} + F_{22'} + F_{33'} = 0. \tag{54}$$

Equations of the kind (53) and (54) arose in the work of Corrigan *et al* on higher-dimensional gauge theories (see [7]).

(d) For each $p \in P$ the four-dimensional subspace

$$C_p = <\Gamma_p, \Gamma_p \Pi^\lambda;\ \lambda \ni p> \tag{55}$$

is a Cartan subalgebra of the Lie algebra $\mathrm{so}(E) = \mathrm{Sk}(E, E)$ which is associated with one of the quadrupoles (46). For example we have, from (50), (55) and (26)

$$C_{0'} = <J_{00'}, J_{11'}, J_{22'}, J_{33'}>. \tag{56}$$

Observe that the Lie algebra $\mathrm{so}(E)$ ($\simeq \mathrm{so}(8)$) consequently possesses a decomposition

$$\mathrm{so}(E) = \oplus_{p \in \mathscr{P}} C_p \tag{57}$$

into seven mutually orthogonal Cartan subalgebras such that

$$[C_p, C_q] = C_{p \cdot q} \qquad p \neq q \in \mathscr{P}. \tag{58}$$

Similarly the Lie subalgebra $V^{21} \simeq$ spin (7) of $\mathrm{so}(E)$ (see (20)) possesses the decomposition

$$V^{21} = \oplus_{p \in \mathscr{P}} B_p \tag{59}$$

into the seven mutually orthogonal Cartan subalgebras

$$B_p = <\Gamma_p \Pi^\lambda;\ \lambda \ni p> \qquad p \in \mathscr{P} \tag{60}$$

which satisfy $[B_p, B_q] = B_{p \cdot q}$ for $p \neq q$.

Appendix

Let E be a $3X8$ algebra, as defined in [1]. Thus E is an eight-dimensional real inner product space which is equipped with a ternary vector cross product $X: E^3 \to E$. By definition $X(a, b, c)$ is linear in each of its arguments, is orthogonal to each of a, b, c and its length $\|X(a, b, c)\|$ equals the volume $\|a \wedge b \wedge c\|$ of the parallelepiped determined by a, b, c (cf [8, 9]). Upon defining the associated ternary multiplication $\{\ \}: E^3 \to E$, as in [1], by

$$\{abc\} = \langle a, b\rangle c + \langle b, c\rangle a - \langle c, a\rangle b + X(a, b, c) \tag{A1}$$

then E is considered also as a $3C8$ algebra, meaning that the ternary composition algebra property

$$\|[abc\}\| = \|a\|\ \|b\|\ \|c\| \tag{A2}$$

is satisfied. The multiplication operators $T_{a,b}$, $\gamma_{a,b}$, $\mu_{a,b}$, and $\sigma_{a,b}$, all $E \to E$, are defined respectively by $c \mapsto X(a, b, c)$, $c \mapsto \{abc\}$, $c \mapsto \{acb\}$ and $c \mapsto \{cba\}$. We list below some of their properties; for further information consult [1].

First note that for $\|e\|=1$ we have $\gamma_{e,e}=\sigma_{e,e}=I$ and $\mu_{e,e}=K$, where K $(=K_e)$ denotes 'conjugation' with respect to the e axis: $Ke=+e$ and $Kx=-x$ for $\langle x, e\rangle=0$. Consequently for $\|a\|=\|b\|=1$ equation (A2) entails that $\gamma_{a,b}$ and $\sigma_{a,b}$ belong to $\mathrm{SO}(E)$ while $\mu_{a,b}$ belongs to $\mathrm{O}_-(E)$. In fact for any a, $b\in E$ we have

$$\gamma_{a,b}\gamma_{b,a}=\|a\|^2\|b\|^2 I=\sigma_{a,b}\sigma_{b,a}. \tag{A3}$$

For $\langle a, b\rangle=0$ we see from (A1) that

$$\gamma_{a,b}=-J_{a,b}+T_{a,b} \qquad \sigma_{a,b}=-J_{a,b}-T_{a,b} \tag{A4}$$

where $J_{a,b}=-2a\wedge b$. Now $T_{a,b}=-T_{b,a}\in \mathrm{Sk}(E, E)\simeq \wedge^2 E$; hence for $\langle a, b\rangle=0$ we also have $\gamma_{a,b}=-\gamma_{b,a}\in \mathrm{Sk}(E, E)$ and $\sigma_{a,b}=-\sigma_{b,a}\in \mathrm{Sk}(E, E)$. Next let $\{b, c, d\}$ be any ordered orthonormal triad, and set $a=\{bcd\}$ and $H=<a, b, c, d>$ (=subspace spanned by a, b, c, d). Then, by lemma B of [1], H is a $3X4$ subalgebra of E, having $\{a, b, c, d\}$ as orthornormal basis, and moreover we have the property

$$-\gamma_{b,a}\gamma_{c,a}\gamma_{d,a}=\Pi^H=+\sigma_{b,a}\sigma_{c,a}\sigma_{d,a} \tag{A5}$$

where $\Pi^H\in \mathrm{SO}(E)$ denotes the involution which equals $+1$ upon the $3X4$ subalgebra H and -1 upon the $3X4$ subalgebra $H^\perp$ (the 'partner' of H).

To make contact with the $d=7$ Clifford algebra considerations in the main part of this paper, fix a unit vector $e\in E$ and let E' denote the seven-dimensional subspace orthogonal to e. Then, using (A3), we have the Clifford algebra properties

$$\Gamma(x)^2=-\langle x, x\rangle I=\Sigma(x)^2 \qquad x\in E' \tag{A6}$$

for the operators $\Gamma(x)$, $\Sigma(x)\in \mathrm{Sk}(E, E)$ defined by

$$\Gamma(x)=\gamma_{x,e} \qquad \text{and} \qquad \Sigma(x)=\sigma_{x,e}. \tag{A7}$$

From (A4) and (A7) and the fact that K $(=K_e)$ commutes with $T_{x,e}$ we have

$$\Sigma(x)=-K\Gamma(x)K \qquad x\in E'. \tag{A8}$$

Let now $\{e_a;\, a=0,1,2,3,0',1',2',3'\}$ be a canonical basis, as defined in [1], for the $3X8$ algebra E, with $e_0=e$. Setting $\Gamma_p=\Gamma(e_p)$, the ordered septet $\{\Gamma_p;\, p=1,2,3,0',1',2',3'\}$ is a set of right-handed anticommuting Clifford operators which satisfy (14) as a special case of the general property (A5). Moreover we have a second septet $\{\Sigma_p\}$ of Clifford operators, where

$$\Sigma_p(\equiv\Sigma(e_p))=-K\Gamma_p K. \tag{A9}$$

From (A5), or from the fact that K commutes with the Π^λ, note that the Σ_p satisfy the 'left-handed' version of (13) and the opposite-sign version of (14) (the latter holding for the same Π^λ as for the Γ_p). From (29), (29′), etc, we have the canonical forms

$$\Sigma_1=J_{01}-J_{23}-J_{3'2'}-J_{1'0'}$$
$$-\Sigma_{23}=\Sigma_1\Pi^\omega=J_{01}-J_{23}+J_{3'2'}+J_{1'0'} \qquad \text{etc.}$$

Remark. Relative to a choice of unit vector $e\in E$ we can define a binary multiplication on E by $ab=\{aeb\}$ and thereby, on account of (A2), view E as the algebra of the real octonions, with e as the identity element. From (A5) we then read off the following property of the left multiplication operators $L_a: b\mapsto ab$ of the octonion algebra: if x, y are orthogonal imaginary octonions then

$$L_x L_y=\Pi^H L_{xy} \tag{A10}$$

where H denotes the quaternion subalgebra $<e, x, y, xy>$. But of course (A5) is more general than (A10), since it holds for any $3X4$ subalgebra $H \subset E$ irrespective of whether or not H happens to contain the octonion identity element e. (Note incidentally, from the preamble to (A5), that *any* three-dimensional subspace of E lies inside a uniquely determined $3X4$ subalgebra of E.)

Remark. In the notation surrounding theorem F of [1], let Θ run through the three involutory outer automorphisms L, M, N of the Lie algebra $\mathrm{Sk}(E, E) = \mathrm{so}(E) \simeq \mathrm{so}(8)$. Then we have the three decompositions

$$\mathrm{so}(E) = V^7_{\Theta=-1} \oplus V^{21}_{\Theta=+1} \tag{A11}$$

where the three subspaces V^7 are spanned respectively by the septets $\{J_{p,e}\}$, $\{\Gamma_p\}$, $\{\Sigma_p\}$ and the three subalgebras V^{21} $(=[V^7, V^7])$ are respectively $\mathrm{so}(E')$, $\mathrm{spin}^R(7)$, $\mathrm{spin}^L(7)$. Moreover the three decompositions (A11) are cyclically permuted (in the order $L \to M \to N \to L$) by the triality automorphism $\Omega = LM = MN = NL$ of $\mathrm{so}(E)$, which last keeps fixed the g_2 subalgebra which is orthogonal to the Γ_p and Σ_p operators, and hence also orthogonal to $J_{p,e} = -(\Gamma_p + \Sigma_p)/2$. The corresponding G_2 subgroup, the common intersection of $\mathrm{SO}(E')$, $\mathrm{Spin}^{\mathrm{R}}(7)$, $\mathrm{Spin}^{\mathrm{L}}(7)$, includes amongst its members the septet $\{\Pi\}^{\lambda}$.

Remark. As is well known, the Lie algebra so(8) is exceptional in possessing a group of outer automorphisms $\{I, L, M, N, \Omega, \Omega^2\}$ of order six (as compared to order two for $\mathrm{so}(2m)$, $m > 4$). One can speculate that just possibly this exceptional feature of so(8) goes along with some exceptional feature of the finite group G_{128} associated, as in (3), with $d = 7$ Clifford algebra. For example, is $d = 7$ Clifford algebra exceptional in that the group G_{128} possesses a subgroup, namely G_{16} of (31), which is a maximal Abelian invariant subgroup consisting entirely of involutions?

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